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XL. *Methodus inveniendi Lineas Curvas ex proprietatibus Variationis Curvaturæ. Auctore Nicolao Landerbeck, Mathes. Profess. in Acad. Upsaliensi Adjuncto. Communicated by Nevil Maskelyne, D. D. F. R. S. and Astronomer Royal.*

Read July 1, 1784.

P A R S S E C U N D A *.

CURVAS, ex proprietate variationis curvaturæ invenire, indice per functionem coordinatarum cujusdam expresso, problema etsi indeterminatum est; juvat tamen ad curvas cognoscendas, quum facile et sponte sese offerunt conditiones determinantes qui rei conveniunt et quæ in casu quovis examini subiecto locum habent. Quo consilio et qua arte calculum inire oporteat, ut et hæc et his affinia peragenda sint, quæ ad curvas ex curvaturæ variatione cognoscendas pertineant, per theoremata quæ sequuntur, exponere conabor.

T H E O R E M A I. (Vide tab. XXI. fig. 2.)

Si curvæ cujusdam LC index variationis curvaturæ fit T, radius curvedinis R, sinus anguli BCD p , posito sinu toto 1, arcus curvæ LC z coordinatæ perpendiculares x et y earumque fluxiones dp , dz , dx , et dy dicantur, erit $\frac{dz}{\int T dz} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam $dx = -Rdp$ et $dz = -\frac{dx}{\sqrt{1-p^2}}$ habetur $\frac{dz}{R} = -\frac{dp}{\sqrt{1-p^2}}$

* See Vol. LXXIII. p. 456.

et quum $dR = Tdz$ erit $R = \int Tdz$ et substitutione $\frac{dz}{\int Tdz} = -$

$$\frac{dp}{\sqrt{1-p^2}}.$$

Cor. 1. Hinc obtinetur $\frac{dx}{R} = -dp$, $\frac{dy}{R} = -\frac{pdp}{\sqrt{1-p^2}}$ et $\frac{dz}{R} = -$
 $\frac{dp}{\sqrt{1-p^2}}.$

Cor. 2. Si Tangens anguli BCD per r , Secans per s designentur habetur $\frac{dz}{\int Tdz} = -\frac{dr}{1+r^2}$ et $\frac{dz}{\int Tdz} = -\frac{ds}{s\sqrt{s^2-1}}.$

Schol. 1. Ex hoc theoremate facilis deducitur methodus generaliter calculandi variationem curvaturæ curvæ cujuscunque. Nam $\int Tdz = -\frac{dz\sqrt{1-p^2}}{dp}$, quantitas vero $\frac{dz\sqrt{1-p^2}}{dp}$ datur, data inter x et y relatione. Sit valor quantitatis $-\frac{dz\sqrt{1-p^2}}{dp} = Z$ functioni curvæ z , $\int Tdz = Z$ et sumtis fluxionibus $Tdz = Zdz$ qua $T = Z$ functioni ipsius z . Si valor quantitatis $-\frac{dz\sqrt{1-p^2}}{dp} = P$ per p expreſſus, erit $\int Tdz = P$ sumtisque fluxionibus $Tdz = Pdp$ et $T = \frac{Pdp}{dz}$, quæ functio est quantitatis p , in potestate semper est $\frac{dp}{dz}$ per p exprimere.

Schol. 2. Hujus etiam theorematibus subsidio inveniri possunt curvæ ex data relatione inter T et z , R et z , R et y , et R et p . Si enim fit $T = Z$ functioni quantitatis z , erit $\int Tdz = \int Zdz + A$, vi theorematibus $\frac{dz}{\int Zdz + A} (= \frac{dz}{\int Tdz}) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{dz}{\int Zdz + A} + C = -\frac{dp}{\sqrt{1-p^2}}$. Posita $\int \frac{dz}{\int Zdz + A} + C = b$ et N nu-

merus cujus logarithmus hyperbolicus 1 habetur $\sqrt{1-p^2} = \frac{N^{b\sqrt{-1}} - N^{-b\sqrt{-1}}}{2\sqrt{-1}}$ et $p = \frac{N^{b\sqrt{-1}} + N^{-b\sqrt{-1}}}{2}$, quæ functiones sunt quantitatis z , quibus positis Z et $\sqrt{1-Z^2}$ respective proveniunt $x (= \int dz \sqrt{1-p^2}) = \int Z dz$ et $y (= \int p dz) = \int dz \sqrt{1-Z^2}$ quarum alterutra curvarum indoles innotescit.

Si $R=X$ functioni abscissæ x provenit $\frac{dx}{X} (= \frac{dx}{R}) = -dp$ et integratione $X (= C - \int \frac{dx}{R}) = p$ unde $\sqrt{1-p^2} = \sqrt{1-X^2}$ et $y (= \int \frac{p dx}{\sqrt{1-p^2}}) = \int \frac{X dx}{\sqrt{1-X^2}}$ æquatio curvæ indolem exprimens.

Et si $R=Y$ functioni ordinatæ y , habetur $\frac{dy}{Y} (= \frac{dy}{R}) = -\frac{p dp}{\sqrt{1-p^2}}$ et integratione $Y (= \int \frac{dy}{Y} + C) = \sqrt{1-p^2}$, unde $p = \sqrt{1-Y^2}$ et $x (= \int \frac{dy \sqrt{1-p^2}}{p}) = \int \frac{Y dy}{\sqrt{1-Y^2}}$ quæ exprimit naturam curvæ.

Hinc colligitur quod quoties $T dz$ perfecte integretur et $\int \frac{dz}{\int Z dz + A}$ obtineatur per arcus circulares dum aut $\int Z dz$ aut $\int dz \sqrt{1-Z^2}$ absolutam admittat integrationem, curvæ erunt rectificabiles, et algebraicæ, si relatio inter x et z vel inter y et z in relationem algebraicam inter x et y permutari possit.

Evidens etiam est quod si X functio est algebraica quantitatis x vel Y quantitatis y , et non solum $\frac{dx}{X}$ vel $\frac{dy}{Y}$ sed etiam $\frac{X dx}{\sqrt{1-X^2}}$ vel $\frac{Y dy}{\sqrt{1-Y^2}}$ quantitates perfecte integrabiles, curvæ evadunt algebraicæ, alias transcendentes.

Exempl.

Exempl. 1. Invenienda fit curva ubi variatio curvaturæ $T =$

$\frac{3 \cdot \sqrt[3]{8a + 27z}^{\frac{2}{3}} - za^{\frac{2}{3}}}{a^{\frac{1}{3}} \sqrt[3]{8a + 27z}^{\frac{2}{3}} - 4a^{\frac{2}{3}}}$. Ut simplicior reddatur calculus ponatur

$\sqrt[3]{8a + 27z}^{\frac{2}{3}} = u$ et $a^{\frac{2}{3}} = b$ erit $z = \frac{u^{\frac{3}{2}} - 8b^{\frac{3}{2}}}{27}$, $dz = \frac{du\sqrt{u}}{18}$, $T = \frac{3u - 2b}{\sqrt{b}\sqrt{u - 4b}}$

et $\int T dz = \frac{u\sqrt{u}\sqrt{u - 4b}}{18\sqrt{b}} + A$; fit constans hæc $A = a$, quod

accidit evanescente $\int T dz u = b$, habetur per theorema

$\frac{du\sqrt{b}}{u\sqrt{u - 4b}} (= \frac{dz}{\int T dz}) = -\frac{dp}{\sqrt{1 - p^2}}$ et integratione $\int \frac{du\sqrt{b}}{u\sqrt{u - 4b}} + C = -\int \frac{dp}{\sqrt{1 - p^2}}$, cujus æquationis termini quum sint arcus circulares

quorum finus $\sqrt{1 - p^2} = \frac{\sqrt{u - 4b}}{\sqrt{u}}$ et cosinus $p = \frac{2\sqrt{b}}{\sqrt{u}}$, posito arcu

constanti $C = 0$, obtinetur $y (= \int p dz) = \int \frac{du\sqrt{b}}{9} + B = \frac{\sqrt{u - 4b}\sqrt{b}}{9}$ nam $B = \frac{4b\sqrt{b}}{9}$, posita $y = 0$ et $u = 4b$, atque $x (=$

$\int dz\sqrt{1 - p^2}) = \int \frac{du\sqrt{u - 4b}}{18} = \frac{u - 4b^{\frac{3}{2}}}{27}$ quibus æquationibus exterminata u et substituta a habetur $y^3 = ax^2$ æquatio pro parabola cubica.

Exempl. 2. Si fit variatio curvaturæ $T = \frac{2z}{a}$ erit $\int T dz (=$

$\int \frac{2z dz}{a}) = \frac{z^2}{a} + A$ et si $Z = 0$ $\int T dz = a$ erit constans $A = a$, atque

vi theorematis $\frac{adz}{a^2 + z^2} (= \frac{dz}{\int T dz}) = -\frac{dp}{\sqrt{1 - p^2}}$ et integratione

$\int \frac{adz}{a^2 + z^2} + C = -\int \frac{dp}{\sqrt{1 - p^2}}$; posito arcu constanti $C = 0$ cæteri

sunt æquales eorumque finus et cosinus, unde $\sqrt{1 - p^2} =$

$\frac{z}{\sqrt{a^2 + z^2}}$, $p = \frac{a}{\sqrt{a^2 + z^2}}$ et $dx (= dz\sqrt{1 - p^2}) = \frac{z dz}{\sqrt{a^2 + z^2}}$ et $dy (=$

$p dz) = \frac{adz}{\sqrt{a^2 + z^2}}$, quibus constat curvam esse catenariam.

Exempl. 3. Sit variatio curvaturæ $T = \frac{a-z}{\sqrt{2az-z^2}}$, evadit $\int T dz$
 $= \sqrt{2az-z^2}$, per theorema $\frac{dz}{\sqrt{2az-z^2}} (= \frac{dz}{\int T dz}) = -\frac{dp}{\sqrt{1-p^2}}$ et
 per integrationem $\int \frac{dz}{\sqrt{2az-z^2}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, si arcus ille
 constans $C = 0$, cæteri sunt æquales eorumque finus et cofinus,
 quo $\sqrt{1-p^2} = \frac{\sqrt{2az-x^2}}{a}$, $p = \frac{a-z}{a}$ et $y (= \int p dz) = \int \frac{a-zdz}{a} =$
 $\frac{2az-z^2}{a}$ æquatio pro cycloide ordinaria.

THEOREMA II.

Manentibus antea adhibitis denominationibus erit $\frac{dx}{y + \int T dx}$
 $= -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam $\frac{dx}{R} = -dp$, erit dividendo per $\sqrt{1-p^2}$, $\frac{dx}{R\sqrt{1-p^2}}$
 $= -\frac{dp}{\sqrt{1-p^2}}$. Propter $1 : \sqrt{1-p^2} :: CD(R) : CF = R\sqrt{1-p^2}$,
 sed $dz : dx :: T dz : T dx$, quæ fluxio est ipsius DE, quare
 $DE = \int T dx$, unde $CF = y + \int T dx$ qua pro $R\sqrt{1-p^2}$ substituta,
 prodit $\frac{dx}{y + \int T dx} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. 1. Quantitas $dy + T dx$ semper est perfecte integrabilis.
 Nam $T dx = -\frac{dx\sqrt{1-p^2}}{dp}$ et $dy = \frac{p dx}{\sqrt{1-p^2}}$ unde $dy + T dx = \frac{p dx}{\sqrt{1-p^2}}$
 $-\frac{dx\sqrt{1-p^2}}{dp}$ et integratione $y + \int T dx = -\frac{dx\sqrt{1-p^2}}{dp}$.

Cor.

Cor. 2. Dicatur femichorda curvaturæ CF F, obtinetur

$$\frac{dx}{F} = -\frac{dp}{\sqrt{1-p^2}}, \quad \frac{dy}{F} = -\frac{pdp}{1-p^2} \text{ et } \frac{dz}{F} = -\frac{dp}{1-p^2}.$$

Cor. 3. Si Tangens anguli BCD per r , Secans per s designentur habetur $\frac{dx}{y + \int Tdx} = -\frac{dr}{1+r^2}$ et $\frac{dx}{y + \int Tdx} = -\frac{ds}{s\sqrt{s^2-1}}$.

Schol. 1. Per hoc theorema via etiam patet calculandi generaliter variationem curvaturæ. Est enim $y + \int Tdx = -\frac{dx\sqrt{1-p^2}}{dp}$, quantitas vero $\frac{dx\sqrt{1-p^2}}{dp}$ datur data inter x et p relatione. Sit valor quantitatis $-\frac{dx\sqrt{1-p^2}}{dp} = X$ functioni abscissæ x æquatione ad curvam inventus, erit $\int Tdx = X - y$ et sumtis fluxionibus $Tdx = \dot{X}dx - dy$, qua $T = \dot{X} - \frac{dy}{dx}$ ubi tam \dot{X} quam $\frac{dy}{dx}$ sunt functiones abscissæ x . Si valor quantitatis $-\frac{dx\sqrt{1-p^2}}{dp} = P$ per p expreſſus, erit $\int Tdx = P - y$ sumtisque fluxionibus $Tdx = \dot{P}dp - dy$, qua $T = \frac{Pdp}{dx} - \frac{p}{\sqrt{1-p^2}}$ ubi $\frac{Pdp}{dx}$ functio est quantitatis p , nam $\frac{dp}{dx}$ per p exprimi poteſt.

Schol. 2. Hoc adhibito theoremate inveniri etiam poſſunt curvæ, ex data relatione inter T et x , F et x , F et y , F et z , et F et p . Poſita enim T functione quantitatis x , patet per curvarum quadraturas, aut perfectam aut imperfectam quantitatis Tdx obtineri integrationem. Sit $\int Tdx = \dot{X} + \int \ddot{X}dx$ functioni vel algebraicæ vel ex parte transcendentis ipsius x , cujus terminis homogeneous valor ipsius $y = \int \ddot{X}dx$ capiatur, iſque ejus indolis ut $\int \sqrt{\ddot{X} + \ddot{X}dx}$, vel quod idem eſt $y + \int Tdx =$

$X + \int \overline{X + X} dx$ integratione absoluta habeatur, permanente $Tdz = Tdx \sqrt{1 - \overline{X}^2}$ perfecte integrabili. Per theorema deinde habetur $\frac{dx}{X + \int \overline{X + X} dx} (= \frac{dx}{y + \int Tdx}) = -\frac{dp}{\sqrt{1-p^2}}$, et per integrationem $\int \frac{dx}{X + \int \overline{X + X} dx} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, si ponatur $\int \frac{dx}{X + \int \overline{X + X} dx} + C = k$ et N basi logarithmorum hyperbolicorum, erit $\sqrt{1-p^2} = \frac{N^k \sqrt{-1} - N^{-k} \sqrt{-1}}{2 \sqrt{-1}}$ et $p = \frac{N^k \sqrt{-1} + N^{-k} \sqrt{-1}}{2}$, $\sqrt{1-p^2}$ et p igitur sunt functiones ipsius x , quæ si ponantur $\sqrt{1 - \overline{X}^2}$ et \overline{X} , habetur $y (= \int \frac{p dx}{\sqrt{1-p^2}}) = \int \frac{\overline{X} dx}{\sqrt{1 - \overline{X}^2}}$, æquatio qua curvæ internoscuntur.

Si fit $F=Y$ functioni quantitatis y erit per Cor. 2. $\frac{dy}{Y} (= \frac{dy}{F}) = -\frac{dp}{1-p^2}$ et integratione $\int \frac{dy}{Y} + \log.C = \log.\sqrt{1-p^2}$, ponatur $\int \frac{dy}{Y} = k$ et N logarithmorum basi, erit facto ad quantitates absolutas transitu $CN^k = \sqrt{1-p^2}$, $p = \sqrt{1 - C^2 N^{2k}}$ et $x (= \int \frac{dy \sqrt{1-p^2}}{p}) = \int \frac{CN^k dy}{\sqrt{1 - C^2 N^{2k}}}$, æquatio quæ indolem curvæ indigat.

Si $F=Z$ functioni ipsius z erit $\frac{dz}{Z} (= \frac{dz}{F}) = -\frac{dp}{1-p^2}$ et integratione $\int \frac{dz}{Z} + \log.C = \log.\sqrt{\frac{1-p}{1+p}}$, et si $\int \frac{dz}{Z} = k$ et N basi logarithmica habetur $p = \frac{1 - C^2 N^{2k}}{1 + C^2 N^{2k}}$ et $y = \int p dz = \int \frac{1 - C^2 N^{2k}}{1 + C^2 N^{2k}} dz$ qua curvæ cognoscuntur.

Constat hinc quod quoties $X + \int \dot{X} dx$ perfecta integratione habeatur $\int \frac{dx}{X + \int \dot{X} dx}$ per arcus circulares dum $\frac{\int \dot{X} dx}{\sqrt{1 - X^2}}$ absolutam admittat integrationem curva fit algebraica, si vero aliter evenierit transcendens.

Quoties $\frac{dy}{Y}$ fit integrale logarithmicum et $\frac{CN^k dy}{\sqrt{1 - C^2 N^{2k}}}$ absolutam admittat integrationem curva est algebraica, in aliis casibus transcendens.

Et quoties $\int \frac{dz}{Z}$ per logarithmos inveniatur, $\frac{1 - C^2 N^{2k} dz}{1 + C^2 N^{2k}}$ absolute fit integrabilis pariter ac $\frac{2CN^k dz}{1 + C^2 N^{2k}}$ curva est algebraica, alias transcendens.

Exempl. 1. Si fit variatio curvaturæ $T = \frac{3 \cdot b^2 - a^2 x \sqrt{a^2 - x^2}}{a^3 b}$ erit $\int T dx (= \frac{a^2 - b^2 \cdot a^2 - x^2 \sqrt{a^2 - x^2}}{a^3 b}) = \frac{a \sqrt{a^2 - x^2}}{ab} - \frac{x^2 \sqrt{a^2 - x^2}}{ab} - \frac{b \sqrt{a^2 - x^2}}{a} + \frac{bx^2 \sqrt{a^2 - x^2}}{a^3}$, si ponatur $y = \frac{b \sqrt{a^2 - x^2}}{a}$ habetur $y + \int T dx = \frac{a^4 + b^2 - a^2 x^2 \sqrt{a^2 - x^2}}{a^3 b}$, adhibendo theorema $\frac{a^3 b dx}{a^4 + b^2 - a^2 x^2 \sqrt{a^2 - x^2}} (= \frac{dx}{y + \int T dx}) = -\frac{dp}{\sqrt{1 - p^2}}$ et integrando $\int \frac{a^3 b dx}{a^4 + b^2 - a^2 x^2 \sqrt{a^2 - x^2}} + C = -\int \frac{dp}{\sqrt{1 - p^2}}$, cujus termini sunt arcus circulares quorum sinus $\sqrt{1 - p^2} = \frac{a \sqrt{a^2 - x^2}}{\sqrt{a^4 + b^2 - a^2 x^2}}$ et cofinus $p = \frac{bx}{\sqrt{a^4 + b^2 - a^2 x^2}}$ evanescente arcu constanti C, quare $y (= \int \frac{p dx}{\sqrt{1 - p^2}}) = \int \frac{b dx}{a \sqrt{a^2 - x^2}} = \frac{b \sqrt{a^2 - x^2}}{a}$ et in hoc casu curva est ellipsis.

Exempl. 2. Sit jam variatio curvaturæ $T = \frac{2\sqrt{2ax+x^2}}{a}$ erit
 $\int Tdx = \frac{x\sqrt{2ax+x^2}}{a} + \int \frac{adx}{\sqrt{2ax+x^2}}$ et posita $y = \int \frac{adx}{\sqrt{2ax+x^2}}$ per-
 fecta integratione habetur $y + \int Tdx = \frac{a+x\sqrt{2ax+x^2}}{a}$. Theore-
 matis itaque auxilio erit $\frac{adx}{a+x\sqrt{2ax+x^2}} (= \frac{dx}{y + \int Tdx} = -\frac{dp}{\sqrt{1-p^2}})$, et
 integratione $\int \frac{adx}{a+x\sqrt{2ax+x^2}} = C = -\int \frac{dp}{\sqrt{1-p^2}}$, si vero arcus ille
 constans $C=0$ cæteri sunt æquales eorumque finus et cosinus,
 unde $\sqrt{1-p^2} = \frac{\sqrt{2ax+x^2}}{a+x}$, $p = \frac{a}{a+x}$ et $y (= \int \frac{pdx}{\sqrt{1-p^2}}) = \int \frac{adx}{\sqrt{2ax+x^2}}$,
 æquatio indicans curvam esse catenariam.

THEOREMA III.

Dicatur cosinus anguli BCD q , posito radio 1, cæterisque
 manentibus denominationibus erit $\frac{dy}{\int Tdy-x} = \frac{dq}{\sqrt{1-q^2}}$.

Est enim $\frac{ay}{R} = aq$, qua per $\sqrt{1-q^2}$ divisa, dat $\frac{dy}{R\sqrt{1-q^2}} = \frac{dq}{\sqrt{1-q^2}}$;
 et ob $1 : \sqrt{1-q^2} :: CD (R) : CG = R\sqrt{1-q^2}$, sed $dz : dy ::$
 $Tdz : Tdy$ cujus integrale est $AE = \int Tdy$, unde $CG (=$
 $AE - AB) = \int Tdy - x$, qua pro $R\sqrt{1-q^2}$ substituta, prodit
 $\frac{dy}{\int Tdy-x} = \frac{dq}{\sqrt{1-q^2}}$.

Cor. 1. Semper $Tdy - dx$ admittit perfectam integrationem.
 Etenim $Tdy = \frac{ddy\sqrt{1-q^2}}{dq}$ et $dx = \frac{qdy}{\sqrt{1-q^2}}$, quibus $Tdy - dx =$
 $\frac{ddy\sqrt{1-q^2}}{dq} - \frac{qdy}{\sqrt{1-q^2}}$ et integratione $\int Tdy - x = \frac{dy\sqrt{1-q^2}}{aq}$.

Cor. 2. Dicatur femichorda curvaturæ CG G, habetur
 $\frac{dy}{G} = \frac{dq}{\sqrt{1-q^2}}, \frac{dx}{G} = \frac{q dq}{1-q^2} \text{ et } \frac{dz}{G} = \frac{dq}{1-q^2}.$

Cor. 3. Dicatur cotangens anguli BCD t , et cofecans v erit
 $\frac{dy}{\int T dy - x} = \frac{dt}{1+t^2} \text{ et } \frac{dy}{\int T dy - x} = \frac{dv}{v \sqrt{v^2-1}}.$

Schol. 1. Quoniam $\int T dy - x = \frac{dy \sqrt{1-q^2}}{dq}$ datur ex data relatione
inter y et q , fit $\frac{dy \sqrt{1-q^2}}{dq} = Y$ functioni ordinatæ y erit $\int T dy =$
 $Y - x$ sumtisque fluxionibus $T dy = \dot{Y} dy - dx$ qua $T = \dot{Y} - \frac{dx}{dy}$
functioni ipsius y . Si autem $\frac{dy \sqrt{1-q^2}}{dq} = Q$ functioni ipsius q
erit $\int T dy = Q - x$ et sumtis fluxionibus $T dy = \dot{Q} dq - dx$, qua
habetur $T = \frac{\dot{Q} dq}{dy} - \frac{q}{\sqrt{1-q^2}}$ per q .

Schol. 2. Hujus theorematis auxilio elicere licet curvas data
relatione inter T et y , G et y , G et x , G et z , et G et q .
Si enim fit T functio ipsius y generaliter $\int T dy = Y + \int \dot{Y} dy + A$,
quæ functio est algebraica ipsius y quoties $\int \dot{Y} dy$ absolute sumi
possit. Assumatur $x = \int \dot{Y} dy$, tali ipsius y functioni ut non
solum $\int T dy - x = Y + \int \dot{Y} + \dot{Y} dy$ sed etiam $\int T dz = \int T dy$
 $\sqrt{1-\dot{Y}^2}$ absoluta integratione habeantur, provenit vi theo-
rematis $\frac{dy}{Y + \int \dot{Y} + \dot{Y} dy + A} (= \frac{dy}{\int T dy - x}) = \frac{dq}{\sqrt{1-q^2}}$ et integratione
 $\int \frac{dy}{Y + \int \dot{Y} + \dot{Y} dy + A} + C = \int \frac{dq}{\sqrt{1-q^2}}.$ Posita $\frac{dy}{Y + \int \dot{Y} + \dot{Y} dy + A}$
 $+ C = l$ et N basi logarithmica erit $q = \frac{N^{l\sqrt{-1}} - N^{-l\sqrt{-1}}}{2\sqrt{-1}}$ et $\sqrt{1-q^2}$

$= \frac{N^{1/2-1} + N^{-1/2-1}}{2}$ quæ functiones sunt quantitatis y , quibus
positis $\overset{||}{Y}$ et $\sqrt{1 - \overset{||}{Y}^2}$ prodit $x (= \int \frac{q dy}{\sqrt{1 - q^2}}) = \int \frac{\overset{||}{Y} dy}{\sqrt{1 - \overset{||}{Y}^2}}$ æqua-
tio quæ indolem curvarum indicat.

Si $G=X$ functioni ipsius x erit per Cor. 2. $\frac{dx}{X} (= \frac{dx}{G}) = \frac{q dq}{1 - q^2}$,
et integratione $\log. CN^l (= \int \frac{dx}{X} + \log. C) = \log. \frac{1}{\sqrt{1 - q^2}}$ si $\int \frac{dx}{X}$
 $= l$, exinde $\sqrt{1 - q^2} = \frac{1}{CN^l}$, $q = \frac{\sqrt{C^2 N^{2l} - 1}}{CN^l}$ et $y (= \int \frac{dx \sqrt{1 - q^2}}{q})$
 $= \int \frac{dx}{\sqrt{C^2 N^{2l} - 1}}$, quæ curvæ naturam indigitat.

Si $G=Z$ functioni ipsius z erit $\frac{dz}{Z} (= \frac{dz}{G}) = \frac{dq}{1 - q^2}$, et integra-
tione $\log. CN^l (= \frac{dz}{Z} + C) = \log. \sqrt{\frac{1+q}{1-q}}$ si $\int \frac{dz}{Z} = l$, unde $q =$
 $\frac{C^2 N^{2l}}{1 + C^2 N^{2l}} \sqrt{1 - q^2} = \frac{2CN^l}{1 + C^2 N^{2l}} x (= \int q dz) = \int \frac{C^2 N^{2l} - 1}{1 + C^2 N^{2l}} dz$ et $y (=$
 $\int dz \sqrt{1 - q^2}) = \int \frac{2CN^l dz}{1 + C^2 N^{2l}}$ quibus curvæ cognoscuntur.

Patet hinc quod quando $Y + \int \overset{||}{Y} dy$ algebraice habeatur

$\int \frac{dy}{Y + \int \overset{||}{Y} dy + A}$ per quadraturam circuli, et $\int \frac{\overset{||}{Y} dy}{\sqrt{1 - \overset{||}{Y}^2}}$ etiam
obtineatur algebraice, curvæ evadunt algebraicæ, secus vero
transcendentes.

Quando $\int \frac{dx}{X}$ vel $\int \frac{dz}{Z}$ obtineatur per logarithmos, et
 $\int \frac{dx}{\sqrt{C^2 N^{2l} - 1}}$, vel tam $\int \frac{C^2 N^{2l} - 1}{1 + C^2 N^{2l}} dz$ quam $\int \frac{2CN^l dz}{1 + C^2 N^{2l}}$ absoluta in-
tegratione, curvæ erunt algebraicæ.

Exempl.

Exempl. 1. Sit index variationis curvaturæ $T = \frac{6y}{a}$ erit $\int T dy = \frac{3y^2}{a} + A$, si quantitas illa constans $A = \frac{a}{2}$ quod evenit quum $\int T dy = \frac{a}{2}$ et $y = 0$; fumatur $x = \frac{y^2}{a}$ erit vi theorematis $\frac{2ady}{a^2 + 4y^2}$ ($= \frac{dy}{T dy - x}$) $= \frac{dq}{\sqrt{1 - q^2}}$ et integratione $\int \frac{2ady}{a^2 + 4y^2} + C = \int \frac{dq}{\sqrt{1 - q^2}}$, cujus æquationis termini quoniam sint arcus circulares quorum sinus $q = \frac{2y}{\sqrt{a^2 + 4y^2}}$ et cosinus $\sqrt{1 - q^2} = \frac{a}{\sqrt{a^2 + 4y^2}}$, arcu constanti $C = 0$, obtinetur $x (= \int \frac{q dy}{\sqrt{1 - q^2}}) = \frac{y^2}{a}$ æquatio pro parabola Apolloniana.

Exempl. 2. Si fit $T = \frac{a^2}{y \sqrt{a^2 - y^2}}$ habetur $\int T dy = \int \frac{dy \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2} + A$, si quantitas illa constans $A = 0$ quod evenit quum $\int T dy = 0$ et $y = a$, et assumatur $x = \int \frac{dy \sqrt{a^2 - y^2}}{y}$, evadit per theorema $-\frac{dy}{\sqrt{a^2 - y^2}}$ ($= \frac{dy}{T dy - x}$) $= \frac{dq}{\sqrt{1 - q^2}}$, et per integrationem $-\int \frac{dy}{\sqrt{a^2 - y^2}} + C = \int \frac{dq}{\sqrt{1 - q^2}}$, quorum arcuum sinus $q = \frac{\sqrt{a^2 - y^2}}{a}$ et cosinus $\sqrt{1 - q^2} = \frac{y}{a}$ si constans ille $C = 0$, atque inde $dx \left(\frac{q dy}{\sqrt{1 - q^2}} \right) = \frac{dy \sqrt{1 - y^2}}{y}$ qua patet curvam esse tractoriam.

T H E O R E M A I V .

Dicatur summa tangentium angulorum HCD et BCD H, et differentia tangentium angulorum HCD et CKB K, retentis reliquis denominationibus erit $\frac{dx}{H dx} = -\frac{dp}{\sqrt{1 - p^2}}$ et $\frac{dy}{K dy} = \frac{dq}{\sqrt{1 - q^2}}$.
Quoniam

Quoniam $dy = \frac{p dx}{\sqrt{1-p^2}}$ erit $dy + T dx = T + \frac{p}{\sqrt{1-p^2}} dx$ et quum $H = T + \frac{p}{\sqrt{1-p^2}}$ habetur $dy + T dx = H dx$. Eodem modo quum $dx = \frac{q dy}{\sqrt{1-q^2}}$ erit $\int T dy - dx = T - \frac{q}{\sqrt{1-q^2}} dy$, sed $K = T - \frac{q}{\sqrt{1-q^2}}$, unde $\int T dy - x = K dy$. Per theorema igitur 2 et 3 provenit $\frac{dx}{\int H dx} = -\frac{dp}{\sqrt{1-p^2}}$ et $\frac{dy}{\int K dy} = \frac{dq}{\sqrt{1-q^2}}$.

Cor. Si fit ut antea tangens anguli BCD r , cotangens t , fecans s , et cofecans v , erit $\frac{dx}{\int H dx} = -\frac{dr}{r+r^2}$ et $\frac{dx}{\int H dx} = -\frac{ds}{s\sqrt{s^2-1}}$, $\frac{dy}{\int K dy} = \frac{dt}{1+t^2}$ et $\frac{dy}{\int K dy} = \frac{dv}{v\sqrt{v^2-1}}$.

Schol. Ope hujus theorematis invenire licet curvas, data relatione inter H et x atque K et y . Itaque fit $H = X$ functioni ipsius x erit $\int H dx = \int X dx + A$, vi theorematis $\frac{dx}{\int X dx + A} (= \frac{dx}{\int H dx}) = -\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int \frac{dx}{\int X dx + A} + C = -\int \frac{dp}{\sqrt{1-p^2}}$. Posita $\int \frac{dx}{\int X dx + A} + C = m$, et N logarithmorum bafi prodit $\sqrt{1-p^2} = \frac{N^{m\sqrt{-1}} - N^{-m\sqrt{-1}}}{2\sqrt{-1}}$ et $p = \frac{N^{m\sqrt{-1}} + N^{-m\sqrt{-1}}}{2}$, quibus functionibus quantitatis x positis $\sqrt{1-X^2}$ et X provenit æquatio $y (= \int \frac{p dx}{\sqrt{1-p^2}}) = \frac{X dx}{\sqrt{1-X^2}}$ naturam curvarum exprimens.

Si $K = Y$ functioni quantitatis y , eadem calculandi ratione habetur $x (= \int \frac{q dy}{\sqrt{1-q^2}}) = \frac{Y dy}{\sqrt{1-Y^2}}$ æquatio qua curvæ cognoscuntur.

Quando

Quando $\int X dx$ vel $\int Y dy$ absoluta integratione, $\int \frac{dx}{\int X dx + A}$ vel $\int \frac{dy}{\int Y dy + A}$ per rectificationem circuli, et $\int \frac{X dx}{\sqrt{1-X^2}}$ vel $\int \frac{Y dy}{\sqrt{1-Y^2}}$ integratione perfecta obtineantur, curva est algebraica.

Exempl. 1. Si fit $H = \frac{a+12x}{2\sqrt{a}\sqrt{x}}$ erit $\int H dx = \frac{a+4x\sqrt{x}}{\sqrt{a}} + A$, et posita $A=0$ habetur per theorema $\frac{dx\sqrt{a}}{a+4x\sqrt{x}} (= \frac{dx}{\int H dx}) = -\frac{dp}{\sqrt{1-p^2}}$ et per integrationem $\int \frac{dx\sqrt{a}}{a+4x\sqrt{x}} + C = \int \frac{dp}{\sqrt{1-p^2}}$, cujus termini quum sint arcus circulares quorum finus $\sqrt{1-p^2} = \frac{2\sqrt{x}}{\sqrt{a+4x}}$ et cofinus $p = \frac{\sqrt{a}}{\sqrt{a+4x}}$, posita $C=0$, obtinetur $y (= \int \frac{p dx}{\sqrt{1-p^2}}) = \sqrt{ax}$, quæ parabolam Apolloniam exprimit.

Exempl. 2. Sit $H = \frac{2a^2-x^4}{ax^2\sqrt{a^2-x^2}}$ erit $\int H dx = \frac{x^2-2a^2\sqrt{a^2-x^2}}{ax} + A$, et si $A=0$, per theorema $\frac{ax dx}{x^2+2a^2\sqrt{a^2-x^2}} (= \frac{dx}{\int H dx}) = -\frac{dp}{\sqrt{1-p^2}}$ et per integrationem $\int \frac{ax dx}{x^2-2a^2\sqrt{a^2-x^2}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, et si $C=0$, $\sqrt{1-p^2} = \frac{\sqrt{a^2-x^2}}{\sqrt{2a^2-x^2}}$, $p = \frac{a}{\sqrt{2a^2-x^2}}$ et $y (= \int \frac{p dx}{\sqrt{1-p^2}}) = \int \frac{adx}{\sqrt{a^2-x^2}}$ æquatio pro curva finuum.

Exempl. 3. Si fit $K = \frac{5a^2+6y^2 \cdot y}{a\sqrt{a^2+y^2}}$ erit $\int K dy = \frac{a^2+2y^2\sqrt{a^2+y^2}}{a^2} + A$, si $A=0$ habetur per theorema $\frac{a^2 dy}{a^2+2y^2\sqrt{a^2+y^2}} (= \frac{dy}{\int K dy}) = \frac{dq}{\sqrt{1-q^2}}$ et integratione $\int \frac{a^2 dy}{a^2+2y^2\sqrt{a^2+y^2}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, qua $q = \frac{y}{\sqrt{a^2+y^2}}$, $\sqrt{1-q^2} = \frac{\sqrt{a^2+y^2}}{\sqrt{a^2+2y^2}}$, si $C=0$, unde $x (= \int \frac{q dy}{\sqrt{1-q^2}}) = \sqrt{a^2+y^2}$ æquatio pro hyperbola æquilatera.

Exempl.

Exempl. 4. Sit $K = \frac{y}{\sqrt{a^2 - y^2}}$ erit $\int K dy = A - \sqrt{a^2 - y^2}$ et si $A = 0$, per theorema $-\frac{dy}{\sqrt{a^2 - y^2}} (= \frac{dy}{\int K dy}) = \frac{dq}{\sqrt{1 - q^2}}$ et per integrationem $-\int \frac{dy}{\sqrt{a^2 - y^2}} + C = \int \frac{dq}{\sqrt{1 - q^2}}$ qua $q = \frac{\sqrt{a^2 - y^2}}{a}$, $\sqrt{1 - q^2} = \frac{y}{a}$ et $dm (= \frac{q dy}{\sqrt{1 - q^2}}) = \frac{dy \sqrt{a^2 - y^2}}{y}$ quæ Tractoriam exprimit.

THEOREMA V.

Designetur productum tangentium angulorum HCD et BCD per U, et angulorum HCD et CKB per V cæteris manentibus erit $\frac{dx}{\int U dx - x} = -\frac{dp}{p}$ et $\frac{dy}{y + \int V dy} = \frac{dq}{q}$.

Quoniam $dy = \frac{p dx}{\sqrt{1 - p^2}}$ et $U = \frac{T p}{\sqrt{1 - p^2}}$ erit $T dy (= \frac{T p dx}{\sqrt{1 - p^2}}) = U dx$, et integratione $\int T dy = \int U dx$ qua $\int T dy - x = \int U dx - x$. Et quoniam $dx = \frac{q dy}{\sqrt{1 - q^2}}$ et $V = \frac{T q}{\sqrt{1 - q^2}}$ erit $T dx (= \frac{T q dy}{\sqrt{1 - q^2}}) = V dy$, $\int T dx = \int V dy$ et $y + \int T dx = y + \int V dy$. Theoremate 2. et 3. prodit $\frac{dx}{\int U dx - x} = -\frac{dp}{p}$ et $\frac{dy}{y + \int V dy} = \frac{dq}{q}$.

Cor. Si anguli BCD tangens, cotangens, &c. designentur ut antea, habetur $\frac{dx}{\int U dx - x} = -\frac{dr}{r \cdot 1 + r^2}$, $\frac{dy}{y + \int V dy} = -\frac{dt}{t \cdot 1 + t^2}$, &c.

Schol. Per hoc theorema curvæ inveniuntur ex data relatione inter U et x, atque inter V et y. Si enim fit $U = X$ functioni ipsius x erit $\int U dx = \int X dx + A$, per theorema $\frac{dx}{\int X dx - x + A} (= \frac{dx}{\int U dx - x}) = -\frac{dp}{p}$, et per integrationem $\int \frac{dx}{\int X dx - x + A} + \log. C =$

log. $\frac{1}{p}$. Ponatur $\int \frac{dx}{Xdx-x+A} = n$ et N basi logarithmica, erit

$$\frac{1}{p} = CN^n, \quad p = \frac{1}{CN^n}, \quad \sqrt{1-p^2} = \frac{\sqrt{C^2N^{2n}-1}}{CN^n} \quad \text{et } y \left(= \frac{pdx}{\sqrt{1-p^2}} \right) = \int \frac{dx}{\sqrt{C^2N^{2n}-1}} \text{ qua æquatione curvarum indoles innotescit.}$$

Si $V=Y$ functioni ipsius y , eadem calculandi ratione provenit $x \left(= \int \frac{qdy}{\sqrt{1-q^2}} \right) = \int \frac{CN^n dy}{\sqrt{1-C^2N^{2n}}}$ qua curvæ cognoscuntur.

Evidens hinc est quod quoties $\int Xdx$ vel $\int Ydy$ algebraice $\int \frac{dx}{Xdx-x+A}$ vel $\int \frac{dy}{y+\int Ydy+A}$ per logarithmos, atque $\int \frac{dx}{\sqrt{C^2N^{2n}-1}}$ vel $\int \frac{CN^n dy}{\sqrt{1-C^2N^{2n}}}$ integratione absoluta, obtineantur, curva est algebraica.

Exempl. 1. Si fit $U=3$ erit $\int Udx = 3x+A$, si vero $\int Udx = \frac{a}{2}$ quando $x=0$ erit $A=\frac{a}{2}$ et $\int Udx-x = \frac{a+4x}{2}$. Per theorema igitur $\frac{2dx}{a+4x} \left(= \frac{dx}{\int Udx-x} \right) = -\frac{dp}{p}$ et per integrationem log. $\sqrt{a+4x} + \log. C = \log. \frac{1}{p}$, posita $p=1$ dum $x=0$ log. $C = -\log. \sqrt{a}$, unde facto a logarithmis transitu $\frac{\sqrt{a+4x}}{\sqrt{a}} = \frac{1}{p}$, qua $p = \frac{\sqrt{a}}{\sqrt{a+4x}}$, $\sqrt{1-p^2} = \frac{2\sqrt{x}}{\sqrt{a+4x}}$ et $y \left(= \int \frac{pdx}{\sqrt{1-p^2}} \right) = \int \frac{dx\sqrt{a}}{2\sqrt{x}} = \sqrt{ax}$ æquatio pro Parabola Apolloniana.

Exempl. 2. Sit $U = \frac{x^3-4a^3}{x\sqrt{x}}$ erit $\int Udx = \frac{x^3-2a^3}{3x^2} + A$, si autem $\int Udx = 0$ et $x=a\sqrt[3]{2}$, erit $A=0$ et $\int Udx-x = \frac{2 \cdot a^3-x^3}{3x^2}$. Vi igitur theorematibus erit $\frac{3x^2dx}{2a^3-x^3} \left(= \frac{dx}{\int Udx-x} \right) = -\frac{dp}{p}$, et integratione

$\log. \frac{a\sqrt{a}}{\sqrt{a^3-x^3}} + \log. C = \log. \frac{1}{p}$ qua $p = \frac{\sqrt{a^3-x^3}}{a\sqrt{a}}$; $\sqrt{1-p^2} = \frac{x\sqrt{x}}{a\sqrt{a}}$

et $y (= \int \frac{p dx}{\sqrt{1-p^2}}) = \frac{dx \sqrt{a^3-x^3}}{x\sqrt{x}}$ æquatio ad curvam quæsitam.

Exempl. 3. Si $V = -\frac{1}{2}$ erit $\int V dy = A - \frac{y}{2}$, posita $\int V dy = 0$ et $y = 0$ erit $A = 0$ et $y + \int V dy = \frac{y}{2}$. Per theorema obtinetur

$\frac{2dy}{y} (= \frac{dy}{y + \int V dy}) = \frac{dq}{q}$ et per integrationem $\log. y^2 + \log. C =$

$\log. q$, si $q = 1$ et $y = a$ erit $\log. C = -\log. a^2$, unde $q = \frac{y^2}{a^2}$,

$\sqrt{1-q^2} = \frac{\sqrt{a^4-y^4}}{a^2}$ atque $dx (= \frac{q dy}{\sqrt{1-q^2}}) = \frac{y^2 dy}{\sqrt{a^4-y^4}}$, curva ergo est Elastica,

Exempl. 4. Sit $V = \frac{a^2-2y^2}{y^2}$ erit $\int V dy = A - \frac{a^2+2y^2}{y}$, si $\int V dy$

$= -3a$ et $y = a$ erit $A = 0$, indeque $y + \int V dy = -\frac{a^2+y^2}{y}$. The-

orematis ope habetur $-\frac{y dy}{a^2+y^2} (= \frac{dy}{y + \int V dy}) = \frac{dq}{q}$ et integratione

$\log. \frac{1}{\sqrt{a^2+y^2}} + \log. C = \log. q$, si $q = 1$ et $y = 0$ erit $\log. C = \log. a$

et exinde $q = \frac{a}{\sqrt{a^2+y^2}}$, $\sqrt{1-q^2} = \frac{y}{\sqrt{a^2+y^2}}$ et $dx (= \frac{q dy}{\sqrt{1-q^2}}) =$

$\frac{ady}{y}$ æquatio pro Logarithmica.

THEOREMA VI.

Dicatur ED L, et AE M, retentis præterea adhibitis denominationibus erit $\frac{dL}{T} = dx$ et $\frac{dM}{T} = dy$.

Quoniam $dz : dx :: Tdz (dR) : Tdx$ habetur $dL = Tdx$ et
S f f 2 dL

$\frac{dL}{T} = dx$. Et quoniam $dz : dy :: Tdz (dR) : Tdy$ obtinetur $dM = Tdy$ et $\frac{dM}{T} = dy$.

Cor. Quum $Tdy = Udx$ et $Tdx = Vdy$, erit substitutione $\frac{dM}{U} = dx$ et $\frac{dL}{V} = dy$.

Schol. Hoc adhibito theoremate inveniri possunt curvæ data relatione inter T et L , T et M , atque inter U et M et V et L . Ponatur $L = T^1$ functioni quantitatis T habetur per theorema $\frac{dT}{T} (= \frac{dL}{T}) = dx$ et integratione $\int \frac{dT}{T} + C = x$ qua T per x datur. Curvæ deinde per theorema 2. elici possunt.

Si $M = T^1$ ipsius T functioni, habetur eodem modo T per y . Si $M = U^1$ functioni ipsius U , obtinetur U per x , et si $L = V^1$ functioni quantitatis V , datur V per y . Per theorema deinde 3. et 5. curvæ inveniuntur.

Evidens quidem est quod curvæ esse non possunt algebraicæ nisi $\int \frac{dL}{T}$, $\int \frac{dM}{T}$, $\int \frac{dM}{U}$ vel $\int \frac{dL}{V}$, obtineantur integratione absoluta.

Exempl. 1. Si fit $L = \frac{aT^3}{54}$ erit $dL = \frac{aT^2 dT}{18}$, et per hoc theorema $\frac{aTdT}{18} (= \frac{dL}{T}) = dx$ et integratione $\frac{aT^2}{36} + C = x$ qua $T = \frac{6\sqrt{x}}{\sqrt{a}}$, si $C = 0$. Per theorema 2. reperitur $y = \sqrt{ax}$, æquatio pro Parabola Apolloniana.

Exempl. 2. Si fit $M = - \int \frac{aT^2 dT}{2 \cdot 1 + T^2}$ erit $dM = - \frac{aT^2 dT}{2 \cdot 1 + T^2}$ et ope theorematis $-\frac{aTdT}{2 \cdot 1 + T^2} (= \frac{dM}{T}) = dy$, et integratione $\frac{a}{4 \cdot 1 + T^2} + C = y$, qua si $C = 0$, $T = \frac{\sqrt{a-4y}}{2\sqrt{y}}$. Per theorema 3. habetur

$dx =$

$dx = \frac{2dy\sqrt{y}}{\sqrt{a-4y}}$, æquatio pro Cycloide ordinaria.

Exempl. 3. Sit $L = -a\sqrt{V}$ erit $dL = -\frac{adV}{2\sqrt{V}}$ et per theorema $-\frac{adV}{2\sqrt{V}} (= \frac{dL}{V}) = dy$ et integratione $\frac{a}{\sqrt{V}} + C = y$, et si $C = 0$, habetur $V = \frac{a^2}{y^2}$ et deinde per theorema 5. $dx = \frac{dy\sqrt{a^2-y^2}}{y}$, qua constat curvam esse Tractoriam.

THEOREMA VII.

Dicatur ut antea CF F et CG G, et summa tangentium angularum HCD et BCD, H, et differentia tangentium angularum HCD et CKB, K, erit $\frac{dF}{H} = dx$ et $\frac{dG}{K} = dy$.

Quoniam $dF (= dy + Tdx) = Hdx$ et $dG (= \int Tdy - x) = Kdy$ provenit $\frac{dF}{H} = dx$ et $\frac{dG}{K} = dy$.

Cor. Quum $F = -\frac{dx\sqrt{1-p^2}}{dp}$ et $G = \frac{dy\sqrt{1-q^2}}{dq}$ provenit divisione $\frac{dF}{FH} = -\frac{dp}{\sqrt{1-p^2}}$ atque $\frac{dG}{GK} = \frac{dq}{\sqrt{1-q^2}}$.

Schol. Auxilio hujus theorematis inveniuntur curvæ ex data relatione inter F et H, G et K, H et p atque K et q. Nam si fit $F = H$ functioni ipsius H, vel $G = K$ functioni ipsius K, habetur per theorema $\frac{dH}{H} (= \frac{dF}{H}) = dx$ et integratione $\int \frac{dH}{H} + C = x$ qua H per x datur. Eodem modo $\frac{dK}{K} (= \frac{dG}{K}) = dy$ et integratione $\int \frac{dK}{K} + C = y$ qua K per y obtinetur. Theorema 4. ulterius progredienti viam monstrat ad curvas inveniendas.

Patet

Patet quod curva non fit algebraica nisi $\int \frac{dH}{H}$ vel $\int \frac{dK}{K}$ obtineantur perfecta integratione.

Exempl. 1. Si fit $F = \frac{a}{\sqrt{1+H^2}}$ habetur per theorema $-\frac{adH}{1-H^2} (= \frac{dF}{H}) = dx$, et integratione $\frac{aH}{\sqrt{1-H^2}} + C = -x$ qua $H = -\frac{x}{\sqrt{a^2-x^2}}$, posita $C=0$. Per theorema deinde 4. provenit $y = \sqrt{a^2-x^2}$ æquatio pro circulo.

Exempl. 2. Sit $F = \frac{a \cdot H^3 + H^2 + 6\sqrt{H^2-12}}{108}$, erit per theorema $\frac{a \cdot H^2 - 6 + H\sqrt{H^2-12} \cdot dH}{36\sqrt{H^2-12}} (= \frac{dF}{H}) = dx$ et integratione facta $\frac{a \cdot H^2 - 6 + H\sqrt{H^2-12}}{72} + C = x$, et posita $C=0$ habetur $H = \frac{a+12x}{2\sqrt{a}\sqrt{x}}$, unde per theorema 4. prodit $y = \sqrt{ax}$ æquatio pro Parabola Apolloniana.

Exempl. 3. Sit $G = -\frac{a \cdot 4 + K^2}{4}$ erit per theorema $\frac{adK}{2} (= \frac{dG}{K}) = dy$, et integratione $\frac{aK}{2} + C = y$, et si $C=0$ $K = \frac{2y}{a}$ unde per theorema 4. $dx = \frac{ady}{y}$, qua constat curvam esse Logarithmicam.

THEOREMA VIII.

Dicatur ut antea productum tangentium angulorum HCD et BCD U, et productum tangentium angulorum HCD et CKB V manentibus reliquis denominationibus erit $\frac{dG}{U-1} = dx$ et $\frac{dF}{1+V} = dy$.

Quoniam

Quoniam $G = \int T dy - x$ erit $dG = T dy - dx$, fed $T dy = U dx$, unde $dG = \overline{U-1} dx$ et $\frac{dG}{U-1} = dx$. Eodem modo quum $F = y + \int T dx$ erit $dF = dy + T dx$, fed $T dx = V dy$ quare $dF = \overline{1+V} dy$ et $\frac{dF}{1+V} = dy$.

Cor. Quoniam $G = \frac{dy\sqrt{1-q^2}}{dq}$ et $F = -\frac{dx\sqrt{1-p^2}}{dp}$, habetur substitutione debita $\frac{dG}{G \cdot U-1} = -\frac{dp}{p}$ et $\frac{dF}{F^2 1+V} = \frac{dq}{q}$.

Schol. Ope hujus theorematis indagantur curvæ data relatione inter G et U vel inter F et V . Nam si fit $G = U$ functioni quantitatis U vel $F = V$ functioni quantitatis V obtinetur per theorema in casu priori $\frac{dU}{U-1} (= \frac{dG}{U-1}) = dx$ et integratione $\int \frac{dU}{U-1} + C = x$, qua U per x habetur; in posteriori $\frac{dV}{1+V} (= \frac{dF}{1+V}) = dy$ et integratione $\int \frac{dV}{1+V} + C = y$, qua V habetur per y . Per theorema deinde 5. curvæ cognoscuntur.

Datur etiam per Cor. U in p , et V in q , et consequenter T in p vel q , nam $U = \frac{Tp}{\sqrt{1-p^2}}$ et $V = \frac{Tq}{\sqrt{1-q^2}}$.

Constat hinc quod curvæ non sint algebraicæ nisi $\int \frac{dU}{U-1}$ vel $\int \frac{dV}{1+V}$ obtineantur integratione absoluta.

Exempl. 1. Si fit $G = \frac{a \cdot \overline{U-3}}{2}$ erit per theorema $\frac{adU}{2U-1} (= \frac{dG}{U-1}) = dx$ et integratione $\log. 1-U + \log. C = \frac{2x}{a}$ et si $C = \frac{a^2}{2}$ log.

log. $\frac{a^2 \cdot \overline{1-U}}{2} = \frac{2x}{a}$ et $\frac{a \cdot \overline{1-U}}{2} = N^{\frac{2x}{a}}$ qua $U = \frac{a^2 - 2N^{\frac{2x}{a}}}{a^2}$. Per theorema deinde 5. habetur $dy = \frac{dx N^{\frac{2x}{a}}}{a}$ qua constat curvam esse Logarithmicam.

Exempl. 2. Si fit $T = \frac{a \cdot \overline{V-1} \sqrt{V+2}}{3\sqrt{3}}$ erit per theorema $\frac{adV}{2\sqrt{3}\sqrt{V+2}} (= \frac{dF}{1+V}) = dy$ et per integrationem $\frac{a\sqrt{V+2}}{\sqrt{3}} = y$ qua $V = \frac{3y^2 - 2a^2}{a^2}$; et per theorema 5. $dx = \frac{dy \sqrt{y^2 - a^2}}{a}$, æquatio ad curvam cujus constructio a quadratura hyperbolæ dependet.

THEOREMA IX.

Sint LC et lc duæ curvæ eandem habentes Evolutam QD, dicatur radiorum osculi CD cD constans differentia cC b , curvæ lc variatio curvaturæ S, ceterisque ut antea manentibus erit $\frac{dR}{R-bS} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam radius curvaturæ DH evolutæ fit $RT = R - bS$, erit $\frac{1}{R-bS} = \frac{1}{RT}$, quæ per $dR (= Tdz) = -\frac{RTdp}{\sqrt{1-p^2}}$ multiplicata, monstrat esse $\frac{dR}{R-bS} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. Si sint ut antea tangens anguli BCD r et secans s , habetur $\frac{dR}{R-bS} = -\frac{dr}{1+r^2}$ et $\frac{dR}{R-bS} = -\frac{ds}{s\sqrt{s^2-1}}$.

Schol. Subsidio hujus theorematism invenire licet curvas, data relatione inter S et R vel inter S et T nam $\frac{S}{T} = \frac{R}{R-b}$. Itaque si
ponatur

ponatur $S = R$ functioni radii curvedinis R , erit $\frac{dR}{R-bR} (= \frac{dR}{R-bS})$

$= -\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int \frac{dR}{R-bR} + C = - \int \frac{dp}{\sqrt{1-p^2}}$. Sit

$\int \frac{dR}{R-bR} + C = f$ et N logarithmorum bafi habetur $\sqrt{1-p^2} =$

$\frac{N^f \sqrt{-1} - N^{-f} \sqrt{-1}}{2\sqrt{-1}}$ et $p = \frac{N^f \sqrt{-1} + N^{-f} \sqrt{-1}}{2}$ functionibus quantitatis

R , quibus R per p exprimi poteft. Per theorema igitur 1. curvas internoscere valemus.

Si $R = S$ functioni quantitatis S habetur $\frac{dS}{S-bS} (= \frac{dR}{R-bS})$

$= -\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int \frac{dS}{S-bS} + C = - \int \frac{dp}{\sqrt{1-p^2}}$, posita

$\int \frac{dS}{S-bS} + C = g$, erit $\sqrt{1-p^2} = \frac{N^g \sqrt{-1} - N^{-g} \sqrt{-1}}{2-\sqrt{1}}$ et $p = \frac{N^g \sqrt{-1} + N^{-g} \sqrt{-1}}{2}$

quibus S per p datur. Per theoremata Partis I. invenire licet curvas omnes eandem evolutam habentes.

Hinc videtur, quod curvæ non fint algebraicæ nifi $\int \frac{dR}{R-bR}$

vel $\int \frac{dS}{S-bS}$ per circuli rectificationem obtineatur.

Exempl. 1. Si fit $S = \frac{2R}{\sqrt{a} \cdot \sqrt{R-a}}$ fupposita $b=a$, erit per

theorema $\frac{dR \sqrt{a}}{2R \sqrt{R-a}} (= \frac{dR}{R-bS}) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione

$\int \frac{dR \sqrt{a}}{2R \sqrt{R-a}} + C = - \int \frac{dp}{\sqrt{1-p^2}}$, fi vero arcus ille constans $C=0$

erit $\sqrt{1-p^2} = \frac{\sqrt{R-a}}{\sqrt{R}}$ qua $R = ap^2$, et per Cor. 1. Theor. 1. ha-

betur $dy = \frac{adx}{\sqrt{x^2-a^2}}$, æquatio pro Catenaria.

Exempl. 2. Sit $S = \frac{5a^2 + R^2}{a \cdot a - 5R}$, posita $b = \frac{a}{5}$ erit per theorema
 $-\frac{adR}{a^2 + R^2} (= \frac{dR}{R - bS}) = -\frac{dp}{\sqrt{1 - p^2}}$ et facta integratione $-\int \frac{adR}{a^2 + R^2} + C$
 $= -\int \frac{dp}{\sqrt{1 - p^2}}$, qua si $C = 0$, habetur $\sqrt{1 - p^2} = \frac{R}{\sqrt{a^2 + R^2}}$ et $R =$
 $\frac{a \sqrt{1 - p^2}}{p}$. Per theorema 1. $dx = \frac{dy \sqrt{a^2 - y^2}}{y}$ qua constat curvam
 esse Tractoriam.



Fig. 1.

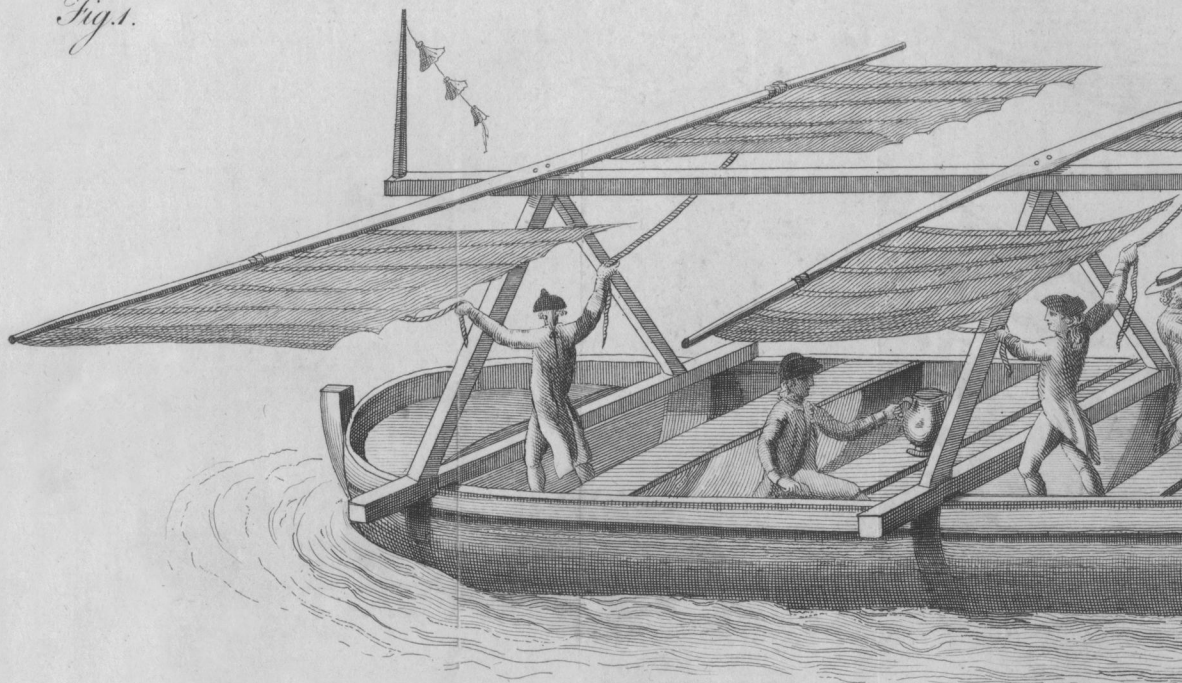


Fig. 2.

